Problem Set 12 – Relativity (`GOI36A`)

08/12/2020

1 Angular Momentum for the Kerr Metric

The Kerr metric for a rotating black hole is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{4GMar\sin^2\theta}{\rho^2} dt\, d\phi + \frac{\rho^2}{\Delta} dr^2 \\ &+ \rho^2\, d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)^2 - a^2\Delta\sin^2\theta\right) d\phi^2 \ , \end{aligned} \tag{1.1}$$

where G is the usual Newton's constant, M and a are both constants, and we have defined the functions ρ and Δ as follows:

$$\begin{aligned} \rho(r,\theta)^2 &= r^2 + a^2\cos^2\theta \ , \\ \Delta(r) &= r^2 - 2GMr + a^2 \ . \end{aligned} \tag{1.2}$$

Intuitively, the parameter M corresponds to the mass of the black hole, while the parameter a corresponds to the angular momentum of the black hole. The goal of this problem is to prove explicitly that a is related to the angular momentum.

1.) First, convince yourself that the the vector $\mathcal{R} = \partial_\phi$ is a Killing vector for the Kerr metric. Then, argue that this means there must therefore be an associated conserved quantity, which we will denote by J. Moreover, argue that J should be the angular momentum of the Kerr solution.

2.) In particular, we will represent the conserved quantity J via a Komar integral of the form

$$J = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\, \sqrt{\gamma^{(2)}}\, n_\mu \sigma_\nu \nabla^\mu \mathcal{R}^\nu \ , \tag{1.3}$$

where Σ is a three-dimensional spacelike hypersurface and $\partial\Sigma$ is its boundary. The vector n is the unit normal to Σ, while σ is the unit normal to $\partial\Sigma$, and the determinant $\gamma^{(2)}$ denotes the determinant of the induced metric on $\partial\Sigma$. Convince yourself that the Komar integral (1.3) makes sense and should compute a conserved quantity.

3.) Compute the integral (1.3) explicitly. You'll first need to specify a spacelike hypersurface Σ and compute the normal vector n. Then, you'll have to find its boundary surface $\partial\Sigma$ and compute the associated normal vector σ. Finally, you have to compute $\nabla^\mu\mathcal{R}^\nu$ and $\gamma^{(2)}$, plug everything into the integral, and evaluate. You should end up with a relation that shows precisely how the parameter a is related to the angular momentum J.

4.) Why have we only considered one component of the angular momentum? Isn't angular momentum supposed to be a vector with three components? What are the other two components for our metric? Be as explicit and precise as you can. You might find that a few equations are helpful in getting your point across.

2 The Ellis Wormhole

A *wormhole* is a type of exotic solution to the Einstein equation that act as a sort of "tunnel" between two points in a spacetime, connecting the points in such a way that going through the wormhole acts as a sort of shortcut. In this problem, we will look at a particularly simple example of a wormhole, known as the *Ellis wormhole*, in order to get a feel for some of their properties.

5.) Consider a four-dimensional spacetime with the metric

$$ds^2 = -dt^2 + d\rho^2 + \left(\rho^2 + n^2\right)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \ , \tag{2.1}$$

with coordinates $x^\mu = (t, \rho, \theta, \phi)$, where n is a real constant and we have set $c = 1$. What is this metric when $n = 0$? When $n = 0$, what is the associated range of the coordinate ρ, and why?

6.) Now, consider letting $n \neq 0$. In the limit where $\rho > 0$ is very large, what does this metric look like? When $\rho < 0$ is very large and very negative, what does this spacetime look like? Is $\rho < 0$ allowed when $n \neq 0$? Why or why not?

7.) Now consider what happens when $\rho = 0$ at an instant $t =$ (const.) in time to make things easier. Compare the situation for $n = 0$ and $n \neq 0$. What is this $\rho = 0$ location physically when $n = 0$? How many dimensions does this particular part of spacetime have? What about when $n \neq 0$, what is this location physically and how many dimensions does this subset of the spacetime have?

8.) Try to give a cartoon drawing of the spacetime for $n \neq 0$, making a smooth drawing connecting what you know from the previous questions about the spacetime at $\rho = 0$, large $\rho > 0$, and large $\rho < 0$. You have (hopefully) just drawn a wormhole!

9.) Can you think of the wormhole we have been describing and drawing so far as an actual "shortcut" between two points in a spacetime? Can you think of a way we might be able to remedy this to actually get a shortcut in spacetime?

10.) In order for the metric (2.1) to actually be physical, it needs to solve the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \ , \tag{2.2}$$

where $T_{\mu\nu}$ is the stress-energy tensor for the as-of-yet-unspecified set of matter fields in our spacetime. Compute the (t, t) component of the left-hand side of the Einstein equation and show that it is given by

$$G_{tt} = -\frac{n^2}{(\rho^2 + n^2)^2} \ , \tag{2.3}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. What does this mean for T_{tt}? What does T_{tt} represent, and is this an acceptable result for it? Why or why not?